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Ordinary Differential Equations on Closed Subsets of Locally Convex Spaces with Applications to Fixed Point Theorems*

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The construction and convergence of an approximate solution to the initial value problem $x' = f(t, x)$, $x(0) = x_0$, defined on closed subsets of locally convex spaces are given. Sufficient conditions that guarantee the existence of an approximate solution are analyzed in relation to the Nagumo boundary condition used in the Banach space case. It is also shown that the Nagumo boundary condition does not guarantee the existence of an approximate solution. Applications to fixed point theorems for weakly inward mappings are given. © 1990 Academic Press, Inc.

I. INTRODUCTION

In this paper we study conditions under which the Cauchy problem

$$x' = f(t, x), \qquad x(0) = x_0 \tag{I.1}$$

can be solved in a sequentially complete locally convex topological vector space (lctvs) X, always assumed Hausdorff. Here, f: $[0, T] \times D \to X$ is continuous with bounded range, $T > 0$, and $D \subseteq X$. Although the natural extension of the Nagumo boundary condition to lctvs does not work (as will be shown by counterexample), we introduce condition (C1) which guarantees the existence of an approximate solution on the set D. At the same time this condition is necessarily satisfied when a solution exists. Next we show that under some additional assumptions (dissipativity condition or compactness-type condition) the approximate solution converges to a solution.

In Section IV we give a detailed analysis of how (C1) can be obtained and its relation to such notions as the weak inwardness and the contingent

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cone. In Section V we prove new fixed point theorems for dissipative and α-condensing maps defined on closed subsets of X. In addition, we provide an example of a weakly inward, contractive, and compact operator defined on a closed bounded convex subset of a Frechet space with no fixed points.

Phillips [1] was the first to study existence theorems under compactness conditions of the non-linear term. Later, Dubinsky [2] obtained a similar result while working in Montel spaces. Millionscikov [3] also obtained some existence theorems using Lipschitz-type conditions. Yuasa [4] gave existence theorems via the measure of non-precompactness. None of them, however, obtained solutions on general subsets of a lctvs X. A recent paper by Agase [5] introduces the concept of an approximate solution. However, Agases condition that guarantees the existence of the approximate solution is given in terms of a metric and seems to be much stronger and more complicated than ours. We should mention that Lemma 2 and Theorems 2, 3, and 4 in [5] may not be true unless the author will assume that non-linear term $f(t, x)$ has a bounded range. Finally, R. S. Hamilton [6], in his expository paper on the inverse function theorem of Nash and Moser, gives a very interesting example of an existence theorem in a Frechet space. His concept of a "smooth Banach map" results in a reduction of the original problem to the Banach space case.

II. Approximate Solution

Let X be a lctvs (real or complex). For each convex and balanced subset $B \subset X$, X_B denotes the vector space spanned by B. We consider X_B with the topology induced by the seminorm p_B, the gauge of B. Consider the Cauchy problem (I.1) for $D \subseteq X$, a closed subset. A function x: $[0, T] \to D$ is said to be a solution of (I.1) if $x(t)$ is continuously differentiable and satisfies (I.1). Problem (I.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds, \qquad t \in [0, T]. \tag{II.1}$$

The integral used in (II.1) is the Riemann integral. Let us point out that Riemann integrability of each continuous function characterizes metrizable, locally convex spaces (see [7, Theorem 3.5.1]). Furthermore (I.1) and (II.1) are not equivalent in non-locally convex spaces. One is referred to [5] for an additional condition on f that guarantees the equivalence of (I.1) and (II.1) in a non-locally convex case.

Let us now assume that (I.1) has a solution $x(t)$ on $[0, T]$ for each $x_0 \in D$. For $z \in D$, $t \in [0, T]$, we have $z + hu \in D$ for $h > 0$ such that $t + h \leqslant T$ and $u = 1/h \int_t^{t+h} f(s, x(s))\, ds$. It follows that for each neighborhood (nbhd) V of zero in X and for each $\varepsilon > 0$ there exists $h \in (0, \varepsilon)$ such that $u \in f(t, z) + V$; or equivalently, there exists $h \in (0, \varepsilon)$ and $z_h \in D$ such that

$$z_h - z - hf(t, z) \in hV. \tag{II.2}$$

When the range of f is bounded, then V in (II.2) can be replaced by a smaller set $B \cap V$, where $B = 2 \operatorname{cl} \Gamma\{f([0, T] \times D)\}$ and ΓA denotes the convex balanced hull of A.

The above considerations suggest that the following conditions are important in the study of differential equations.

Condition C1. For each balanced and convex nbdh V of zero, there exists a bounded and balanced subset B of X such that for each $\varepsilon > 0$, $x \in D$, and $t \in [0, T]$ we can find $0 < h \leqslant \varepsilon$ and $x_h \in D$ satisfying $x_h - x - hf(t, x) \in h(V \cap B)$.

Condition C2. For each balanced and convex nbhd V of zero, each $\varepsilon > 0$, $x \in D$, and $t \in [0, T]$, we can find $0 < h \leqslant \varepsilon$ and $x_h \in D$ satisfying $x_h - x - hf(t, x) \in hV$.

We have at once the following proposition:

PROPOSITION II.1. *Let us assume that a continuous function f: $[0, T] \times D \to X$ has bounded range. Then* (C1) *is a necessary condition for the existence of a solution to* (I.1) *for each $x_0 \in D$.*

We note that (C1) and (C2) are equivalent when X is locally bounded (i.e., there exists a bounded nbhd of zero), for example, when X is a normed space. We remark also that when D is convex, $(x_h - x)/h - f(t, x) \in B \cap V$ implies that $(z_k - x)/k - f(t, x) \in B \cap V$ for all $0 < k \leqslant h$, where $z_k = k((x_h - x)/h) + x = (k/h)x_h + (1 - k/h)x \in D$.

Now we will prove the main result of this section.

THEOREM II.2. *Let X be a lctvs that is sequentially complete and D be a closed subset of X. In addition, let us assume that*

(a) *f: $[0, T] \times D \to X$ is continuous and the range of f is bounded*

(b) *Condition* C1 *is satisfied.*

Then for each $x_0 \in D$, each abs convex (i.e., convex and balanced) nbhd V of zero, and each $\varepsilon > 0$, there exists a (V, ε)-approximate solution $x_{(V,\varepsilon)}(t)$ on $[0, T]$ to the problem (I.1); *i.e., there exists $\{t_i\}$ in $[0, T]$ with $t_0 = 0$, $t_i - t_{i-1} \leqslant \varepsilon$, $i = 1, 2, ..., p$, and $t_p = T$, such that*

(i) *$x_{(V,\varepsilon)}(0) = x_0$ and $x_{(V,\varepsilon)}(t) - x_{(V,\varepsilon)}(s) \in (t - s)[(V \cap B) + B_f]$ for some abs convex bounded subset B_f such that $f([0, T] \times D) \subset B_f$;*

(ii) $x_{(V,\varepsilon)}(t_{i-1}) \in D$ *and* $x_{(V,\varepsilon)}(t)$ *is linear on* $[t_{i-1}, t_i]$ *for each* $i = 1, 2, ..., p$;

(iii) *if* $t \in (t_{i-1}, t_i)$, *then* $(d/dt)\, x_{(V,\varepsilon)}(t) - f(t_{i-1}, x_{(V,\varepsilon)}(t_{i-1})) \in V \cap B$;

(iv) *if* $t, s \in [t_{i-1}, t_i]$, $y \in D$ *with* $y - x_{(V,\varepsilon)}(t_{i-1}) \in (t_i - t_{i-1})[(V \cap B) + B_f]$, *then* $f(t, y) - f(s, x_{(V,\varepsilon)}(t_{i-1})) \in V \cap (2B_f)$.

Proof. By increasing B if necessary, we can assume that (C1) holds when $(V \cap B)$ is replaced by $\frac{1}{3}(V \cap B)$ and B is a closed abs convex subset of X such that $6B_f \subset B$ and $x_0 \in B$. For simplicity, we omit (V, ε) as a subscript. Our proof is by induction on i. If $x(t)$ is defined on $[0, t_{i-1})$, $t_{i-1} < T$ and (i)–(iv) are satisfied with $[0, T]$ replaced by $[0, t_{i-1}]$, then choose $\delta_i \in [0, \varepsilon]$ such that δ_i is the supremum of δ with $t_{i-1} + \delta \leqslant T$ and the following properties:

(1) if $t, s \in [t_{i-1}, t_{i-1} + \delta]$, $y \in D$ with $y - x(t_{i-1}) \in \delta[(V \cap B) + B_f]$, then $f(t, y) - f(s, x(t_{i-1})) \in V \cap (2B_f)$,

(2) $x_\delta - x_{i-1} - \delta f(t_{i-1}, x_{i-1}) \in \delta(B \cap V)$ for some $x_\delta \in D$, where $x_{i-1} = x(t_{i-1})$.

Then $\delta_i > 0$ because $t_{i-1} < T$, f is continuous, B and B_f are bounded sets, and (C1) is satisfied. Therefore, we can choose $0 < \delta_i/2 \leqslant h_i < \delta_i \leqslant \varepsilon$ such that (1) and (2) are satisfied with δ replaced by h_i. Now we define $t_i = t_{i-1} + h_i$ and $x(t_i) = x_{h_i}$.

For $t \in [t_{i-1}, t_1]$ we define

$$x(t) = \frac{(t - t_{i-1})\, x(t_i) + (t_i - t)\, x(t_{i-1})}{t_i - t_{i-1}}.$$

Properties (i)–(iv) follow easily with T replaced by t_i. To complete the proof we need to show that $t_p = T$.

Suppose it is not true, i.e., $t_i < T$ for all $i \geqslant 1$. Let $v = \lim t_i$. By (i), $\{x(t_i)\}$ is Cauchy in X and in X_B, hence, convergent to $z \in D$. We observe that $x(t_i) \to z$ also in the space X_B. Indeed, X_B is normed by p_B because B, as a bounded subset of X, does not contain any straight line. But then $x_0 \in B$ and closedness of B in X imply that $x(t_i) \to z$ in X_B. Now continuity of f together with the fact that $x(t_i) \to z$ in X_B guarantees that there exist $0 < \delta_0 \leqslant \varepsilon$ and $i_0 \geqslant 2$ such that (1) is satisfied for $i \geqslant i_0$. Furthermore, there exist $i_1 \geqslant i_0$ and $0 < \delta \leqslant \delta_0$ such that for $x_i = x(t_i)$, $i \geqslant i_1$, and some $x_\delta \in D$ we have $(x_\delta - x_{i-1})/\delta - f(t_{i-1}, x_{i-1}) = (x_\delta - z)/\delta - f(v, z) + (z - x_{i-1})/\delta + f(v, z) - f(t_{i-1}, x_{i-1}) \in \frac{1}{3}(V \cap B) + \frac{1}{3}(V \cap B) + (\frac{1}{3}V) \cap 2B_f \subset V \cap B$, where the last inclusion follows from the fact that $6B_f \subset B$. Finally, $h_i \to 0$ implies $\delta_i \to 0$. On the other hand, δ_i is the supremum of all $\delta > 0$ such that (1) and (2) are satisfied. Hence $\delta_i \geqslant \delta$, which obviously cannot be true for i large enough. We have reached a contradiction.

The above result can be localized in D. Indeed, let f be such that $f([0, T] \times D_w) \subset B_f$ for some bounded subset B_f of X, where $D_w = D \cap (x + W)$ and W is a fixed closed abs convex nbhd of zero in X. Next, assume the following local version of (C1):

> For each abs convex nbhd of zero V there exists a bounded and balanced subset B of X such that for each $\varepsilon > 0$, $x \in D_w$, and $t \in [0, T]$ we can find $0 < h \leqslant \varepsilon$ and $x_h \in D$ satisfying $x_h - x - hf(t, x) \in h(V \cap B)$.

Then we can obtain an approximate solution $x_{(V,\varepsilon)}(t) \in x_0 + W$ for T such that $T(V + B_f) \subset W$ and for all $V \subset (\frac{1}{2}) W$. For the existence of such $T > 0$, the boundedness of B_f is essential.

Let us note that Theorem I.1 holds if the sequential completeness of X is replaced by the fast (or Mackey) completeness of X (each bounded set in X is contained in a bounded and abs convex subset $B \subset X$ such that X_B is complete). Each sequentially complete space is fast complete. (See [8, Prop. II.1.9, p. 33].) On the other hand, when D is compact, no completeness assumption is needed. Furthermore, $\{x_{(V,\varepsilon)}(t)\}$ can be considered as a net of continuous functions from $[0, T]$ into X (into D when D is convex) where V runs through the nbhds of zero in X and $\varepsilon > 0$. We define the following order: $(V, \varepsilon) \geqslant (W, \eta)$ if $V \subset W$ and $\varepsilon \leqslant \eta$. We have

COROLLARY II.3. *Suppose that the assumptions of Theorem* I.1 *are satisfied. If, in addition,* $x_{(V,\varepsilon)}(t) \to x(t)$ *for* $t \in [0, T]$, *then* $x(t)$ *is a solution to* (I.1).

III. EXISTENCE THEOREMS

In this section we present existence theorems for the Cauchy problem (I.1) under suitable dissipativity and compactness conditions on the nonlinear map f. We begin by studying dissipativity. Let us assume the existence of a family $\mathscr{H}$ of functions that satisfies the following properties (see [9, p. 41] for the Banach space analog):

(H1) Each $H \in \mathscr{H}$ is a continuous function from $[0, T] \times D \times D$ into $\mathbb{R}_+$ such that $H(t, x, x) \equiv 0$. Also, $H(t, x, y) > 0$ for some $H \in \mathscr{H}$ if $x, y \in D$, $x \neq y$, $0 \leqslant t \leqslant T$.

(H2) For each $H \in \mathscr{H}$ there exists a continuous seminorm p and $L_{p,H} > 0$ such that $|H(t, x, y) - H(t, x_1, y_1)| \leqslant L_{p,H}[p(x - x_1) + p(y - y_1)]$ for every $x, x_1, y, y_1 \in D$ and $t \in [0, T]$.

(H3) If $\{x_\alpha\}$ and $\{y_\alpha\}$ are nets in D with the same index set such that $H(t, x_\alpha, y_\alpha) \to 0$ for each $H \in \mathscr{H}$, then $x_\alpha - y_\alpha \to 0$.

(H4) For each $H \in \mathscr{H}$, $D_{-}H(t, x, y) \overset{\text{def}}{=} \liminf_{h \to 0^+} (1/h)[H(t, x, y) - H(t-h, x-hf(t, x), y-hf(t, y))] \leqslant g_H(t, H(t, x, y))$ for $t \in [0, T]$ and $x, y \in D$, where g_H: $[0, T] \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function such that $g_H(t, 0) \equiv 0$ and such that $u(t) \equiv 0$ is the unique solution to $u' = g_H(t, u)$, $u(0) = 0$.

We have:

THEOREM III.1. *Assume that*

(i) f: $[0, T] \times D \to X$ *is a continuous function with bounded range where* D *is a convex closed subset of a complete lctvs* X.

(ii) *Condition* C1 *is satisfied.*

(iii) *There exists a family of Liapunov functions,* $\mathscr{H}$, *with the properties* (H1)–(H4).

Then there exists a unique solution to (I.1) *on* $[0, T]$, *for each* $x_0 \in D$.

Proof. The convexity of D implies that $x_{(V,\varepsilon)}(t) \in D$ for $t \in [0, T]$. Let $H \in \mathscr{H}$ be an arbitrary element of $\mathscr{H}$. For such $\mathscr{H}$ let p be a continuous seminorm as in (H2) and let $r > 0$. For $0 < \varepsilon$, $\eta < r$, and V, $W \subset \{x: p(x) \leqslant r\}$ abs convex nbhds of zero in X, let $m(t) = H(t, x_{(V,\varepsilon)}(t), x_{(W,\eta)}(t))$. Using (H2), properties (iii) and (iv) of the approximate solution, and (H4) we obtain $D_{-}m(t) \leqslant g_H(t, m(t)) + 4L_{p,H}r$, for all but a finite number of $t \in [0, T]$. To complete the proof we proceed in a standard way. Since $m(0) = 0$, an application of (H4) implies that $H(t, x_{(V,\varepsilon)}(t), x_{(W,\eta)}(t)) \to 0$ uniformly on $[0, T]$. Since H is an arbitrary element of $\mathscr{H}$, (H3) implies that $\{x_{(V,\varepsilon)}(t)\}$ is a uniformly Cauchy net on $[0, T]$. Completeness of X together with Corollary II.2 imply that (I.1) has a solution on $[0, T]$. By using (H1) uniqueness follows in a standard way.

Let X be a lctvs with $D \subset X$ and $\mathscr{P} = \{p\}$ a family of seminorms that generate the locally convex topology of X. For f: $D \to X$ we say that f is ω-dissipative ($\omega = \{\omega_p\}$) if for each $p \in \mathscr{P}$ there are constants $\omega_p \in \mathbb{R}$ such that $p(x - y - h(f(x) - f(y))) \geqslant (1 - h\omega_p)\, p(x - y)$ for $x, y \in D$ and $h > 0$. If $\omega_p = 0$ for all $p \in \mathscr{P}$ then f is called dissipative. This definition agrees with the concept of dissipativity in a normed space. Furthermore, we say that f: $D \to X$ is locally bounded if for each $x \in D$ there exists a neighborhood W of zero in X such that $f[D \cap (x + W)]$ is bounded.

Following the lines of the proof of Theorem 6.1 in [10, pp. 247–248], applied to each $p \in \mathscr{P}$, and utilizing Theorem III.1, we have

THEOREM III.2. *Suppose* X *is a complete lctvs* X. *Let* D *be a closed and convex subset of* X *and* f: $D \to X$ *be a continuous map which is* ω*-dissipative* *and locally bounded. Then* (I.1) *has a solution* $x_z(t)$ *on* $[0, \infty)$ *for each* $z \in D$ *if Condition* C1 *is satisfied. Also,*

$$p(x_z'(t)) \leqslant p(f(z)) \exp(\omega_p t),$$
$$p(x_z(t) - x_w(t)) \leqslant p(z-w) \exp(\omega_p t),$$

for $p \in \mathscr{P}$, z, $w \in D$, $t \geqslant 0$.

We have the following generalization of the Banach space result (see [10, Lemma 6.4, p. 251]).

COROLLARY III.3. *Suppose that the assumptions of Theorem* III.1 *are satisfied with* $\omega_p < 0$ *for each* $p \in \mathscr{P}$. *Then there is a unique* $z \in D$ *such that* $f(z) = 0$.

Now we shall consider compactness criteria on the non-linear term f which lead to existence results. Let M be the set of all bounded subsets of X, and $\mathscr{A}$ be the set of functions a: $\mathscr{P} \to [0, \infty)$ with the topology of pointwise convergent and the natural partial ordering: $a_1 < a_2$ if $a_1(p) \leqslant a_2(p)$ for each $p \in \mathscr{P}$. Following Sadovskii [11], we define the Kuratowski measure of non-compactness on X, generated by the family $\mathscr{P}$ of seminorms, as the function α: $M \to \mathscr{A}$, defined by the formula $[\alpha(\Omega)](p) = \inf\{d > 0$: the set Ω can be split into finitely many subsets whose diameters with respect to the seminorms p are not greater than $d\}$. Properties of the Kuratowski measure of non-compactness may be found in [11], particularly Theorem 1.2.3 of the quoted reference.

The following lemma is a mean value theorem in lctvs.

LEMMA III.4. *Let* x: $[0, T] \to X$ *be left-side differentiable*; *i.e., the quotient* $(x(t) - x(t-h))/h$ *tends to a limit* $D_x(t)$ *when* $h \to 0^+$ *for each* $0 < t \leqslant T$. *Then*

$$\left\{\frac{x(t) - x(t-h)}{h}: t \in [0, T],\ 0 < h < t\right\} \subset \operatorname{cl} \operatorname{conv}\{D_- x(t): t \in (0, T]\}.$$

For a proof we refer either to Colombeau [12, Theorem 1.3.2, p. 55] or to Deimling [13, Proposition 2.1, p. 2]. The proof in [13] is done in the normed space case. However, the method extends with no change to a general locally convex space setting.

THEOREM III.5. *Let* X *be a complete and metrizable lctvs* X. *Suppose that Condition* C1 *is satisfied, as well as the following conditions*:

(i) f: $[0, T] \times D \to X$ *is a continuous function with bounded range, where* D *is a closed subset of* X.

(ii) *$\alpha(f(I, \Omega))(p) \leqslant g_p(\alpha(\Omega)(p))$ for all continuous seminorms p and bounded $\Omega \in D$, where $I = [0, T]$ and, for each p, g_p: $\mathbb{R}_+ \to \mathbb{R}$ is a continuous function with $g_p(0) = 0$ and the property that the initial value problem $u' = g_p(u)$, $u(0) = 0$ has only the trivial solution $u \equiv 0$.*

Then there is a solution to (I.1) *on* $[0, T]$ *for each* $x_0 \in D$.

Proof. We start by choosing (V_n, ε_n)-approximating solutions $x_n(t)$ where $V_{n+1} \subset V_n$ is a countable base of nbhds of zero in X and $\varepsilon_n \downarrow 0^+$. Let p be a continuous seminorm and $r > 0$. For $k \geqslant N \geqslant l$ (N is such that $\varepsilon_n \leqslant r$ and $V_N \subset \{x: p(x) \leqslant r\}$), let $m(t) = \alpha(\Omega_k(t)(p)$, where $\Omega_k(t) = \{x_n(t): n \geqslant k\}$ and $m(0) = 0$. Furthermore, $m(t)$ is continuous because of the property (i) approximate solutions and the subadditivity of α. We have

$$D_- m(t) \stackrel{\text{def}}{=} \liminf_{h \to 0^+} \frac{1}{h} [m(t) - m(t-h)]$$
$$\leqslant \alpha \left[\left\{ \frac{x_n(t) - x_n(t-h)}{h} : n \geqslant k \right\} \right] (p).$$

Lemma III.4 and convexity properties of α give us $D_- m(t) \leqslant \liminf_{h \to 0^+} \alpha(\Omega'_k(I_h)](p)$, where $I_h = [t-h, t]$, $\Omega'_k(I_h) = \bigcup_{t \in I_h} \Omega'_k(t)$, and $\Omega'_k(t) = \{D_- x_n(t): n \geqslant k\}$. Also, for $k \geqslant N$,

$$\alpha[\Omega'_k(I_h)](p) \leqslant \alpha \left[\bigcup_{t \in I_h} \{f(\tau_n(t), x_n(\tau_n(t))): n \geqslant k\} \right] (p) + 2r,$$

where we have used property (iii) of approximate solutions. Assumption (ii) of the theorem gives us $\alpha[\Omega'_k(I_h)](p) \leqslant g_p[\alpha(\Omega_k(I_h))] + 2r$. By the continuity of g_p and the uniform continuity of α, $g_p[\alpha(\Omega_k(I_h))](p) \to g_p[\alpha(\Omega_k(t))](p)$ as $h \to 0^+$. Therefore,

$$D_- m(t) \leqslant g_p(m(t)) + 2r$$

for $t \in [0, T]$. The rest of the proof proceeds in the same manner as in the proof of the corresponding theorem in the Banach space setting (see [9, Theorem 2.5.1, pp. 45–46] or [13, Theorem 2.1, pp. 21–22]).

In utilizing compactness arguments in a non-metrizable lctvs, a slightly stronger condition than (C1) is required. In effect, the choice of the bounded set B must be uniform with respect to neighborhoods.

Condition C3. There exists a bounded and abs convex $B \subset X$ such that for each abs convex nbhd V of zero, each $\varepsilon > 0$, and each $(t, x) \in [0, T] \times D$ we can find $0 < h \leqslant \varepsilon$ and $x_h \in D$ satisfying $x_h - x - hf(t, x) \in h(V \cap B)$.

We recall some basic topological notions. We say that X satisfies the Mackey condition (see [14, p. 177]) if each convergent sequence $\{x_n\}$ is bournologically convergent (i.e., there exists a bounded and balanced subset $B \subset X$ and a sequence $r_n \to 0$ such that $x_n \in r_n B$ for $n \geqslant 1$). We say that X satisfies the strict Mackey condition if for every bounded and abs convex subset A of X, there exists a bounded and abs convex $B \supset A$ such that on A, the topology induced by X or X_B is the same. We note that each metrizable lctvs satisfies both of these conditions. For other examples of such spaces see [14, pp. 177–178 and 182–184].

THEOREM III.6. *Let X be a quasi-complete lctvs which satisfies the strict Mackey condition. Suppose that* (i) *and* (ii) *of Theorem* III.5 *are satisfied, as well as Condition* C3. *Then there is a solution to* (I.1) *on* $[0, T]$ *for each* $x_0 \in D$.

Proof. The strict Mackey condition implies the existence of an abs convex and bounded subset $B' \supset B$ such that the topology on B induced by X or $X_{B'}$ is the same. Therefore, Condition C3 implies that Condition C2 is satisfied in $X_{B'}$. It is now clear that for $\varepsilon > 0$ we can construct ε-approximate solutions with the properties (i) to (iv), where $V \cap B$ and V in (iv) are replaced by $\varepsilon B'$. By taking $\varepsilon_n \to 0$ we obtain the corresponding sequence of approximate solutions $x_n(t)$ to (I.1). Moreover, $\{x_n(t)\} \subset X_B$ and is uniformly bounded there. Now we proceed in the same way as in the proof of Theorem III.5 to obtain that $\mathrm{cl}\{x_n(t)\}$ is compact in X. Again the strict Mackey condition gives that $\mathrm{cl}\{x_n(t)\}$ is compact in $X_{B'}$. As before, Corollary II.3 completes the proof.

The strict Mackey condition on Theorem III.6 is superflous if (C3) is replaced by the condition that (C2) is satisfied in some X_B where B is a closed bounded abs convex subset of X containing D and $f(D)$.

IV. CASES OF EQUIVALENCE

In this section we shall indicate certain cases when the conditions (C1) and (C2) are equivalent. In fact, in many situations the direct applications of the Nagumo condition (C2) is rather difficult. However, the formulation of Condition C1 may be well suited for these applications. Therefore, when it is known that the conditions are equivalent, the application of (C1) may provide an important tool in obtaining existence results. Indeed, we shall see this shortly in constructing fixed point theorems.

Let us introduce few definitions. For a subset $D \subset X$ of a lctvs X and $x \in D$, define the contingent cone to D at x by $T_D(x) = \bigcap_V \bigcap_{\varepsilon > 0} \bigcup_{0 < h \leqslant \varepsilon} ((1/h)(D - x) + V)$, where V runs through abs convex nbhds of zero in X. The contingent cone can be viewed as a generalization of a tangent space in the following sense: If D is a smooth manifold embedded in $\mathbb{R}^n$; then $T_D(x)$ is the tangent space to D at x. More facts about $T_D(x)$ can be found in [15, 16].

For $p_\alpha \in \mathscr{P}$, a family of seminorms on X that determines the locally convex topology of X, let us define $\operatorname{dist}_\alpha(x, D) = \inf_{u \in D} p_\alpha(x-u)$, the distance function from $x \in X$ to D in terms of p_α. For $x \in D$ we define the inward set $I_D(x) = \{x + c(u-x): u \in D \text{ and } c \geqslant 0\}$. We say that a mapping $f: D \to X$ is inward in case $f(x) \in I_D(x)$ for each $x \in D$. f is weakly inward in case $f(x) \in \operatorname{cl} I_D(x)$ for each $x \in D$. The notion of a weakly inward map was introduced by Halpern and Bergman [17] in the context of locally convex spaces, and plays an important role in the fixed point theory. However, they used it successfully only when D was compact. Later it was used for subsets of normed spaces. We already noted that when X is normed (C1) and (C2) are equivalent. Below we show that the same is true when D is compact and convex. Furthermore, at the end of Section V, we give an example of a weakly inward, contractive, and compact operator (defined on a closed, bounded, and convex set D) that does *not* have fixed points.

All the results below are true if $f(t, x)$ depends explicitly on t. However, to simplify our notation we consider the case where f does not depend on t.

We have the following standard result (see, for example, Lemma 1.5 of [24]).

PROPOSITION IV.1. *Let D be a subset of a lctvs X and $f: D \to X$. Then the following statements are equivalent*:

(a) (C2) *is satisfied.*

(b) *$f(x) \in T_D(x)$ for each $x \in D$.*

(c) $\liminf_{\lambda \to 0^+} (1/\lambda) \operatorname{dist}_\alpha(x + \lambda f(x), D) = 0$ *for each $p_\alpha \in \mathscr{P}$ and $x \in D$.*

If, in addition, D is convex, then (a), (b), *and* (c) *are equivalent to*

(d) *$x + f(x) \in \operatorname{cl} I_D(x)$ for each $x \in D$ (i.e., $I + f$ is weakly inward on D)*

(e) *$f(x) \in \operatorname{cl}[\bigcup_{h>0} (1/h)(D-x)]$ for each $x \in D$.*

Let $(X, T) = \operatorname{ind}(X_n, T_n)$; i.e., X is an inductive limit of an increasing sequence of locally convex spaces. Then $(X, T) = \operatorname{ind}(X_n, T_n)$ is called regular if for each bounded subset $B \subset X$ there exists $n_1 > 1$ such that $B \subset X_{n_1}$ and B is bounded in (X_{n_1}, T_{n_1}). Also, (X, T) is said to be strongly boundedly retractive if given $n \geqslant 1$ there exists $k > n$ such that for each bounded subset B of (X, T) contained in X_n, the topologies T and T_k coincide on B. Floret [18] showed that for $(X, T) = \operatorname{ind}(X_n, T_n)$, where $X_n \subset X_{n+1}$ and X_n are normed spaces for $n \geqslant 1$, (X, T) is strongly boundedly retractive if and only if (X, T) is regular and satisfies the Mackey condition.

We shall now indicate some cases when (C2) is equivalent to (C1). Throughout, D is a subset of the lctvs X and $f\colon D \to X$.

THEOREM IV.2. *Conditions* C1 *and* C2 *are equivalent in each of the following cases*:

(1) *D is convex and $x + f(x) \in I_D(x)$ for each $x \in D$ (i.e., $I + f$ is inward).*

(2) *D is compact and convex and f is continuous.*

(3) *X has the weak topology of a normed space and D is convex.*

(4) *X satisfies the strict Mackey condition, D is weakly compact and convex, and f is uniformly continuous with bounded range.*

(5) *X is the inductive limit of an increasing sequence of Banach spaces, it is fast complete, and it satisfies the Mackey condition. In addition, D and $f(D)$ are bounded subsets of X.*

Proof. The proof for Case (1) is straightforward. For (2), let V be an open nbhd and $\varepsilon > 0$. For each $x \in D$ there exist $0 < h \leqslant \varepsilon$ and $x_h \in D$ such that $x_h - x - hf(x) \in hV$. For $x \in D$, $N(x) = \{y \in D\colon x_h - y - hf(y) \in hV\}$ is a non-empty open nbhd of x in D. Moreover, $D = \bigcup_{x \in D} N(x)$. Compactness of D implies that there exists $x_1, ..., x_k \in D$ such that $D = \bigcup_{i=1}^{k} N(x_i)$. Let us define

$$B = \bigcup_{i=1}^{k} \left(\frac{x_{h_i} - D}{h_i} - f(D) \right).$$

Then B is the required bounded (even relatively compact) subset of X in (C1). The remark after Proposition II.1 completes the proof for Case (2).

By Proposition IV.1 and the fact that $T_D(x)$ is convex when D is convex (see [5, pp. 405–408]), we can write

$$\begin{aligned} \text{weak cl}\left[\bigcup_{h>0} \frac{1}{h}(D-x) \right] &= \text{norm cl}\left[\bigcup_{h>0} \frac{1}{h}(D-x) \right] \\ &= \bigcap_{r>0} \bigcap_{\varepsilon>0} \bigcap_{0<h\leqslant\varepsilon} \left(\frac{1}{h}(D-x) + rB \right), \end{aligned}$$

where B denotes the norm-closed unit ball in X. This shows that (C2) implies (C1) and, at the same time, gives the equality

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda} \operatorname{dist}_X(x + \lambda f(x), D) = 0, \tag{IV.1}$$

where $\operatorname{dist}_X(z, D)$ denotes the distance from z to D in X viewed as a normed space.

Given Case (4), we shall show that there exists a bounded abs convex subset B of X with $D \cup f(D) \subset B$, such that on D the weak topology of X and the weak topology of X_B coincide. In particular, D is weakly compact in X_B. Indeed, let B be a bounded and abs convex subset of X that is given in the definition of the strict Mackey condition of X. Clearly, X^*, the dual of X, is contained in X_B^*. Therefore, on D the weak topology of X_B is stronger than the weak topology of X. Furthermore, the fact that for a convex subset D of a lctvs Y $(= X_B)$, the fact of weak compactness depends only on the topology induced by Y (not Y weak) on D (see [14, Chap. 2, Sect. 8, Ex. 2). Hence D is weakly compact in X_B. Moreover, every coarser Hausdorff topology on D coincides with the weak topology of X_B on D, which demonstrates the assertion.

Since every net $\{y_\alpha\}_{\alpha \in I}$ such that $y_\alpha = r_\alpha x_\alpha \to x \in X$ where $0 < r_\alpha \to \infty$ and $x_\alpha \in D$ contains a bounded subnet of $\{y_\alpha\}$, we can find $x_\alpha \in D$ and $h_\alpha \to 0$ such that $z_\alpha = (x_\alpha - x)/h_\alpha - f(x) \to 0$ and $\{z_\alpha\}$ is a bounded net. Next, because $x_\alpha \to x$, continuity of f implies that $y_\alpha = (x_\alpha - x)/h_\alpha - f(x_\alpha) = z_\alpha + f(x) - f(x_\alpha) \to 0$ and $\{y_\alpha\}$ is a bounded net. Let V be an abs convex nbhd of zero in X and $\varepsilon > 0$. For each $x \in D$, there is an abs convex and bounded subset B_x of X such that $y_\alpha \to 0$ in X_{B_x}. By increasing B_x, if necessary, we can assume that $B \subset B_x$. Let W be a nbhd of zero in X such that $W + W \subset V$. By uniform continuity, there exists a balanced nbhd U of zero in X such that $y \in D$ and $y - x \in U$ imply $f(y) - f(x) \in W$. Take $0 < \eta \leqslant \varepsilon$ and $\delta_x > 0$ such that $\eta f(D) + \eta \delta_x B_x \subset U$ and $\delta_x B_x \subset W$. Let us note that η can be chosen independently of $x \in D$. There exists α_0 such that for $\alpha \leqslant \alpha_0$ we have $h_\alpha \leqslant \eta$ and $(x_\alpha - x)/h_\alpha - f(x_\alpha) \in \delta_x B_x$. Now, if V_w is a weakly open nbhd of zero in X_B, then

$$x \in h_{\alpha_0} f(x_{\alpha_0}) - x_{\alpha_0} + h_{\alpha_0} \delta_x(B_x + V_w) \cap B_x = N_x$$

and N_x is a weakly open nbhd of x in (D, weak topology of X_B). By the above consideration, D is weakly compact in X_B, and therefore $D \subset \bigcup_{i=1}^n N_{x_i}$ for some $n \geqslant 1$. For $B_0 = \bigcup_{i=1}^n \delta_{x_i} B_{x_i}$ and each $x \in D$ we have $(x_h - x)/h - f(x_h) \in B_0 \cap W$ for some $0 < h \leqslant \eta \leqslant \varepsilon$ and $x_h \in D$. But $x_h - x \in hf(D) + hB_0 \subset U$ implies $f(x_h) - f(x) \in W$. Hence we have

$$\frac{x_h - x}{h} - f(x) \in B_0 \cap W + (2B_0) \cap W \subset (B_0 + 2B_0) \cap (W + W).$$

Finally, the remark after Proposition II.1 completes the proof for Case (4).

By the result of Qiu Jing Huei quoted in [19], X is fast complete if and only if X is regular. Hence X is strongly boundedly retractive. It also follows that D, $f(D) \subset X_{n_0}$ for some $n_0 \geqslant 1$ and are bounded there. Since every net $\{y_\alpha\}_{\alpha \in I}$ such that $y_\alpha = r_\alpha x_\alpha \to x \in X$ where $0 < r_\alpha \to \infty$ and $x_\alpha \in D$ contains a bounded subnet of $\{y_\alpha\}$, we can find $x_\alpha \in D$ and $h_\alpha \to 0$ such that $y_\alpha = (x_\alpha - x)/h_\alpha - f(x) \to 0$ and $\{y_\alpha\}$ is a bounded net in X. Then there exists $n_1 \geqslant n_0$ such that $y_\alpha \to 0$ in X_{n_1}, and we can choose $B = B_{n_1}$ in (C1), where B_{n_1} is the unit ball in X_{n_1}. We remark that B does not depend on V. This completes the proof of the theorem.

We note that an important class of spaces satisfying (5) of Theorem IV.2 is the duals (with the strong topology) of quasi-normable and metrizable spaces (see [14, Theorem 1, p. 165; Theorem 5, p. 174; and Corollary 2, p. 175] and Exercise 2.1, p. 177 of [18]).

V. Fixed Point Theorems and Examples

As an application of Theorem I.1 we obtain new fixed point theorems for lctvs, which were known previously only in the Banach space setting.

Theorem V.1. *Suppose X is quasi-complete (closed and bounded sets are complete) lctvs X; D is a closed, convex, and locally bounded subset of X; and $f\colon D \to X$ is a continuous and locally bounded map which is ω-dissipative with $\omega_p < 1$ for each $p \in \mathscr{P}$. If Condition C1 is satisfied with f replaced by $f - I$, then there is a unique $z \in D$ satisfying $f(z) = z$.*

Proof. $f - I$ is ω-dissipative with $\omega = \{\omega_p - 1\}$ and $\omega_p - 1 < 0$ for all $p \in \mathscr{P}$. By Corollary III.3, there is a unique $z \in D$ such that $f(z) - z = 0$; i.e., z is a unique fixed point of f.

Let us note that f is ω-dissipative with $\omega_p < 1$ if f is a contraction, i.e., $p(f(x) - f(y)) \leqslant c_p\, p(x-y)$ for all $p \in \mathscr{P}$, $x, y \in D$, and $0 \leqslant c_p < 1$. Using the results of Theorem IV.2(4) and IV.2(5), we have the following corollaries:

Corollary V.2. *Suppose that X is a quasi-complete lctvs with the strict Mackey condition. Assume that D is a convex and weakly compact subset of X and $f\colon D \to X$ is a contraction map. If f is weakly inward on D then f has a unique fixed point.*

Corollary V.3. *Suppose that X is the inductive limit of increasing sequence of Banach spaces X_n and, in addition, X is quasi-complete and satisfies the Mackey condition. Assume that D is a closed and locally bounded subset of X and $f\colon D \to X$ is a contraction. If f is weakly inward on D then f has a unique fixed point.*

We observe that each metrizable lctvs satisfies the strict Mackey condition. In addition, when X is reflexive, each bounded subset of X is relatively weakly compact. Corollary V.3 generalizes the classical result that in a Banach space each weakly inward contraction defined on a closed subset D has a unique fixed point. Later in the section we show that, in general, Theorem V.1 may not be true if (C1) for $f-I$ is replaced by the weak inwardness of f on D.

If α is the Kuratowski measure of non-compactness defined in Section III, we say that $f\colon D \to X$ is α-Lipschitz if $[\alpha(f(B))](p) \leqslant k_p[\alpha(B)](p)$ for all bounded $B \subset D$, $p \in \mathscr{P}$, and for some $k_p \geqslant 0$. f is called α-condensing if $[\alpha(f(B))](p) < [\alpha(B)](p)$ whenever $[\alpha(B)](p) > 0$. In [17] Bergman and Halpern proved the following fixed point theorem (for a simple proof of this theorem, see Theorem 1.3 of [24]):

THEOREM V.4 (Theorem 4.1 of [17]). *Let X be a topological vector space such that continuous linear functionals distinguish points (for example, every lctvs). Let D be a compact convex subset of X and $f\colon D \to X$ be a continuous weakly inward map (equivalently,* (C2) *is satisfied for $f-I$). Then f has a fixed point.*

The conditions (C1) and (C2) are equivalent under the assumption of the above theorem (D is convex and compact and f is continuous). We wish to generalize Theorem V.4 to the case when D is not necessarily compact but f is α-condensing. Our first preliminary result uses a method developed by Deimling [20, Theorem 1, pp. 69–70].

PROPOSITION V.5. *Suppose X is a quasi-complete lctvx X. Let D be a closed bounded convex subset of X and $f\colon D \to X$ be a continuous α-condensing map with bounded range. Then f has a fixed point in D if $f-I$ satisfies* (C2) *for some X_B, where B is a closed bounded abs convex subset of X containing D and $f(D)$.*

Proof. All conditions are invariant under translation, so we may assume $0 \in D$. It is enough to assume that f is α-Lipschitz with $k = k_p < 1$. Indeed, that $f-I$ satisfies (C2) in X_B implies $kf-I$ satisfies (C2) in X_B (see (d) of Proposition IV.1). Now if $k_n f(x_n) = x_n \in D$ for $k_n < 1$ and $k_n \to 1$ then $[\alpha(\{x_n\}\colon n \geqslant 1)](p) = 0$ for each $p \in \mathscr{P}$ if f is α-condensing. Since X is quasi-complete, this means that $\mathrm{cl}\{x_n\}$ is compact in X. Hence $x_{n_\alpha} \to x_0 = f(x_0) \in D$ for some subnet of $\{x_n\}$.

We want to show that the problem can be reduced to the situation of Theorem V.4. Let us choose $w \in D$, $0 < v_n < 1$, $\sum_{n=1}^{\infty} v_n < \infty$ and consider $D_0 = D$, $D_n = \mathrm{cl}\,\mathrm{conv}[(f(D_{n-1}) + v_n B) \cup \{w\} \cap D_{n-1}$, $n \geqslant 1$. It is clear that $D_n \subset D_{n-1}$; $D_n \neq \varnothing$ ($w \in D_n$) for $n \geqslant 1$; and each D_n is closed, convex, and bounded. Furthermore, for each $p \in \mathscr{P}$, we have

$[\alpha(D_n)](p) \leqslant [\alpha(f(D_{n-1}))](p) + v_n[\alpha(B)](p) \leqslant k^n[\alpha(D)](p) + [\alpha(B)](p)[k^{n-1}v_1 + \cdots + kv_{n-1} + v_n]$. Hence $[\alpha(D_n)](p) \to 0$ as $n \to \infty$ for each $p \in \mathscr{P}$. Therefore, $D^* = \bigcap_{n>0} D_n$ is non-empty, convex, and compact. Our next task is to show that f is weakly inward on D^*. In the process of doing that we first show that f is weakly inward on D_n, $n \geqslant 0$ as a map from D_n to X_B. It is true for $n=0$. If it is true for D_{n-1}, fix $0 < \varepsilon \leqslant v_n$ and $x \in D_n \subset D_{n-1}$. There is $0 < h \leqslant \varepsilon$ and $x_h \in D_{n-1}$ such that $x_h - x - h(f(x) - x) \in h\varepsilon B$. Let $u = x + (1/h)(x_h - x)$. Then $u - f(x) \in B$, i.e., $u \in f(D_{n+1}) + v_n B$. Moreover, $(1-h)x + hu = x_h \in D_{n-1}$ implies that $x_h \in D_n$. To show that f is weakly inward on D^*, let $x \in D^* \subset D_n$, $n \geqslant 0$, and consider the initial value problem $u_n' = f(u_n) - u_n$, $u_n(0) = x$ on $[0, T]$, $T \geqslant 0$. By the remark after Theorem III.6 there is a solution $u_n(t) \in D_n$, $t \in [0, T]$, for every $n \geqslant 0$. Furthermore, for each $t \in [0, T]$, $\{u_n(t)\}$ is relatively compact in X and $\{u_n(\cdot)\}$ is equicontinuous (D and $f(D)$ are bounded sets). By Ascoli's theorem for topological vector spaces, there is a subnet of $\{u_n(t)\}$ converging uniformly on $[0, T]$ to $u(t) \in D^*$ and $u(t)$ is a solution to $u' = f(u) - u$, $u(0) = x$ on $[0, T]$. This implies that $f - I$ satisfies (C2) in X (in fact, (C1) holds too since D^* and $f(D^*)$ are bounded sets). Equivalently, f is weakly inward on D^* in X. Finally, Theorem V.4 completes the proof.

THEOREM V.6. *Assume that*

(i) *X is a quasi-complete metrizable lctvs X; D is a closed, bounded, and convex subset of X; and f: $D \to X$ is a continuous and α-condensing map with bounded range.*

(ii) *$f - I$ satisfies* (C1).

Then f has a fixed point in D.

Proof. Theorem III.5 implies that the initial value problem $x' = f(x) - x$, $x(0) = x_0 \in D$ has a solution in D. Thus, we conclude that $f - I$ satisfies (C3). Next, X being metrizable satisfies the strict Mackey condition, therfore $f - I$ is weakly inward in some X_B with a bounded B containing D and $f(D)$. Now, Proposition V.5 completes the proof.

As we mentioned before, in some situations (C1) and (C2) are equivalent. This is so when D is a convex, weakly compact subset of a metrizable lctvs X and f is uniformly continuous with bounded range (Theorem IV.2). We have:

COROLLARY V.7. *Suppose that* (i) *of Theorem* V.6 *holds but X is not necessary quasi-complete. In addition, assume that D is weakly compact and f is weakly inward and uniformly continuous. Then f has a fixed point in D.*

D, because it is weakly compact, is a complete subset of X with the weak topology. Since closed convex subsets of X are weakly closed, it follows that D is a complete subset of X. Quasi-completeness of X in Theorem III.5 was only needed for the subsets of D.

Theorem V.6 and Corollary V.7 are also true if the assumption that f is α-condensing is replaced by the condition that f is b-condensing [21]. Theorem V.6 extends Reich's result [21, Theorem 3.3]. He assumed that f is inward on D. Furthermore, in [22] Reich conjectured that for every continuous weakly inward α-contraction f: $D \to X$, where D is a bounded closed convex subset of a Frechet space X, the initial value problem $x' = f(x)$, $x(0) = x_0 \in D$ has a solution on $[0, T]$, $T > 0$. The example given below shows that this is not true in general. The result in [22] given in the remark after the theorem is not true if K is not assumed to be closed in each E_{p_n}, $n > 1$.

We are not able to prove a non-metrizable analog of Theorem V.6. The difficulty is to show the convergence of a net (not a sequence) of approximate solutions to a solution of the initial value problem $x' = f(x) - x$, $x(0) = z \in D$. Let us note that this problem does not occur when f is ω-dissipative, since in the process of showing the convergence of a net of approximate solutions we show that our net is Cauchy. When f is α-Lipschitz, one would like to show that the net of approximate solutions is relatively compact, but this may not be true even if the net is convergent. However, we do have the following corollary:

COROLLARY V.8. *Assume that* (i) *of Theorem* V.6 *holds, with X not necessarily metrizable. In addition, suppose that X is an inductive limit of an increasing sequence of Banach spaces and satisfies the strict Mackey conditions. Then f has a fixed point in D if f is weakly inward on D.*

Proof. By Theorem IV.2(5), Condition C3 holds for some bounded and abs convex subset B of X. Since X satisfies the strict Mackey condition, an application of Proposition V.5 completes the proof.

Finally, let us present two examples for which (C2) is satisfied and (C1) is not satisfied, and for which there exist neither fixed points nor a solution of the initial value problem.

For $D \subset X$, a point $x \in D$ is a local conical support point of D if there is a cone C with non-empty interior and a nbhd V_x of x such that $(x + C) \cap V_x \cap D = \{x\}$. A point $x \in D$ of a real lctvs X is called a support point of D if there exists a nonzero continuous linear functional F on X such that $\sup F(D) = F(x)$. The functional F is called a support functional of D. When D is convex, $x \in D$ is a local conical support point of D if and only if x is a support point of D. This fact easily follows from the separation theorem. The following result relates Condition C2 to the notion of local supports points (see also Proposition 4.1 of [25]).

PROPOSITION V.9. *Let D be a subset of a lctvs X. Suppose that $x \in D$ and there is a $w \in X$ such that we can find $\varepsilon > 0$ and a nbhd V of zero in X such that for each $0 < h \leqslant \varepsilon$ and each $z \in D$, $x + hw - z \in X \backslash hV$. Then x is a local conical support point of D.*

Proof. Let U be a closed abs convex nbhd of zero in X satisfying $2U \subset V$. Note that $w \in X - 2U$. Consider $C = \bigcup_{h>0} \{y \in X\colon hw - y \in hU\}$. C is a cone with non-empty interior. For p the gauge of U, and $r = \min(1, \varepsilon)$, define $N = \{x\colon p(x) \leqslant r\}$ a nbhd of zero in X. To prove that x is a local conical support point it is enough to show that $(x + C \cap N) \cap D = \{x\}$. Suppose $0 \neq y \in C$, $p(y) \leqslant r$, and $x + y \in D$. For $h > 0$ such that $hw - y \in hU$ we have $hp(w) - p(y) \leqslant h$. Hence $h \leqslant p(y)/(p(w) - 1) \leqslant 2p(y)/p(w) \leqslant r \leqslant \varepsilon$, where we used the fact that $p(w) \geqslant 2$. Therefore $x + hw - (x + y) \in X - hV$. On the other hand, $x + hw - (x + y) = hw - y \in hU \subset (\frac{1}{2})\, hV$. This contradiction completes our proof.

As a first example, let us consider the set D given by Peck [23] in Theorem 4. D is a closed bounded convex set without support points in some Frechet space X. Proposition V.9 gives us that for each $\varepsilon > 0$, for each balanced nbhd V of zero in X, and for each $x \in D$ and $w \in X$ we can find $0 < h \leqslant \varepsilon$ and $x_h \in D$ such that $x_h - x - hw \in hV$. This is exactly Condition C2 for $f \equiv w$. Using Proposition IV.1(e) we can also say that $\mathrm{cl}[\bigcup_{h>0} (1/h)(D - x)] = X$. In other words, (C2) is trivially satisfied for each $x \in D$ and any function $f\colon D \to X$. For example, if $f(x) = c$ for some $c \in X \backslash D$ and all $x \in D$ then f is dissipative and weakly inward on D, yet f has no fixed points in D.

As a second example, with D as above, let $w \in X \backslash D$ and $z \in D$. We claim that (C1) does not hold on D for $f(x) \equiv w - z$. Indeed, the initial value problem $x' = f(x)$, $x(0) = z$, $0 \leqslant t \leqslant 1$, does not have an approximate solution on $[0, 1]$ in spite of the fact that (C2) holds on D for $f(x) = w - z$, $x \in D$. Indeed, even if Condition (iv) is dropped and $B \cap V$ is replaced by V in (i)–(iii) of Theorem II.2, there fails to be any "approximate solution" in this weaker sense.

REFERENCES

1. R. S. PHILLIPS, Integration in convex linear topological spaces, *Trans. Amer. Math. Soc.* **47** (1940), 114.
2. E. DUBINSKY, Differential equations and differential calculus in Montel spaces, *Trans. Amer. Math. Soc.* **110** (1964), 1.
3. V. M. MILLIONSCIKOV, A contribution to the theory of differential equations $dx/dt = f(x, t)$ in locally convex spaces, *Soviet Math. Dokl.* **1** (1960), 288.

4. T. Yuasa, Differential equations in a locally convex space via the measure of nonprecompactness, *J. Math. Anal. Appl.* **84** (1981), 534.
5. S. B. Agase, Existence and stability of ordinary differential equations in locally convex spaces, *Nonlinear Anal. Theory Meth. Appl.* **5** (1981), 713.
6. R. S. Hamilton, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.* **7** (1982), 65.
7. S. Rolewicz, "Metric Linear Spaces," Reidel, Dordrecht, 1985.
8. M. De Wilde, "Closed Graph Theorems and Webbed Spaces," Pitman, London, 1978.
9. V. Lakshmikantham and S. Leela, "Nonlinear Differential Equations in Abstract Spaces," Pergamon, Oxford, 1981.
10. R. H. Martin, "Nonlinear Operators and Differential Equations in Banach Spaces," Wiley, New York, 1976.
11. B. N. Sadovskii, Limit-compact and condensing operators, *Russian Math. Surveys* **27** (1982), 85.
12. J. F. Colombeau, "Differential Calculus and Holomorphy," North-Holland, Amsterdam, 1982.
13. K. Deimling, Differential equations in Banach spaces, *in* "Lecture Notes in Mathematics," Vol. 596, Springer-Verlag, Berlin, 1977.
14. A. Grothendieck, "Topological Vector Spaces," Gordon & Breach, New York, 1973.
15. J. P. Aubin and I. Ekeland, "Applied Nonlinear Analysis," Wiley, New York, 1984.
16. J. P. Aubin and A. Cellina, "Differential Inclusions," Springer-Verlag, Berlin, 1984.
17. B. R. Halpern and G. M. Bergman, A fixed-point theorem for inward and outward maps, *Trans. Amer. Math. Soc.* **130** (1968), 353.
18. K. Floret, Folgenretraktive Sequenzen Lokalkonvexer Räume, *J. Reine Angew. Math.* **259** (1973), 65.
19. J. Kucera and K. McKennon, Köthe's example of an incomplete LB-space, *Proc. Amer. Math. Soc.* **93** (1985), 79.
20. K. Deimling, Fixed points of condensing maps, in Volterra equations, *in* "Lecture Notes in Mathematics," Vol. 737, Springer-Verlag, Berlin, 1979.
21. S. Reich, Fixed points in locally convex spaces, *Math. Z.* **125** (1972), 17.
22. S. Reich, A fixed point theorem for Frechet spaces, *J. Math. Anal. Appl.* **78** (1980), 33.
23. N. T. Peck, Support points in locally convex spaces, *Duke Math. J.* **38** (1971), 271.
24. S. Reich, On fixed point theorems obtained from existence theorems for differential equations, *J. Math. Anal. Appl.* **54** (1976), 26.
25. S. Reich, Approximate selections, best approximations, fixed points, and invariant sets, *J. Math. Anal. Appl.* **62** (1978), 104.